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On (m, n)-twin semigroups

by

Sándor LAJOS

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A semigroup S is called *(m, n)-twin* if, for every couple a, b of elements of S, we have $a^mSa^n=b^mSb^n$ for a pair m, n of positive integers. A semigroup S is called *archimedean* if $a^n \in SbS$ for every couple a, b of elements of S and for some positive integer n.

N. Kuroki [1] has studied (1, 1)-twin semigroups. His results will be generalized in this short note.

THEOREM 1. *Any (m, n)-twin semigroup is archimedean.*

Proof. Let S be an (m, n)-twin semigroup, $a, b \in S$. Then we have

$$a^{m+n+2} \in Sa^mSa^n=Sb^mSb^n \subseteqq SbS ,$$

that is, S is archimedean.

THEOREM 2. *Any (m, n)-twin semigroup has at most one idempotent element.*

Proof. Let S be an (m, n)-twin semigroup, $E(S)$ the set of all idempotents of S. Let $e, f \in E(S)$. Then we have

$$e=e^3 \in eSe=e^mSe^n=f^mSf^n=fSf .$$

Hence it follows that there is an element x in S such that

$$e=fxf .$$

Similarly

$$f=eye ,$$

where $y \in S$. Thus we conclude

$$e=fxf=fxf^2=ef=e(eye)=f .$$

Hence $E(S)$ contains at most one element.

THEOREM 3. *For an (m, n)-twin semigroup S the following conditions are equivalent*:

(A) *S is a group.*

(B) *S is a monoid.*

(C) *S is an (m, n)-regular semigroup.*

Proof. Evidently, (A) implies (B). Assume (B), and let e be the identity element of S. Then, for any element a of S, we have

$$a=eae\in eSe=a^mSa^n\ ,$$

that is, a is (m, n)-regular, indeed.

Now assume (C), i.e. let S be (m, n)-regular, $a\in S$, and B an (m, n)-ideal of S. Then, for any $b\in B$, we have

$$a\in a^mSa^n=b^mSb^n\subseteqq B^mSB^n\subseteqq B\ ,$$

whence it follows that $S\subseteqq B$, that is, $S=B$.

Thus S has no proper (m, n)-ideal. Hence it follows that S is a group (see [2]).

THEOREM 4. *A group is an (m, n)-twin semigroup.*

Proof. Suppose that S is a group with the identity e and $a, b\in S$. Then we have

$$a^mSa^n=ea^mSa^ne=(b^mb^{-m})a^mSa^n(b^{-n}b^n)\subseteqq b^mSb^n\ .$$

Similarly we obtain

$$b^mSb^n\subseteqq a^mSa^n\ ,$$

that is,

$$a^mSa^n=b^mSb^n$$

holds for every couple a, b of elements of S. Therefore S is (m, n)-twin.

COROLLARY 5. *For a regular semigroup S the following conditions are equivalent*:

(A) *S is (m, n)-twin.*

(B) *S is a group.*

We show that there exists an (m, n)-twin semigroup which is not $(1, 1)$-twin.

Example 1. The semigroup $S=\{0, 1, 2, 3, 4, 5\}$ with the multiplication table

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	1	1
3	0	0	0	1	0	1
4	0	0	1	1	2	2
5	0	0	1	0	2	2

is a (2, 1)-twin semigroup because $a^2Sa=\{0\}$ for each element a of S. It fails to be (1, 1)-twin because $1S1=\{0\}\neq 5S5=\{0, 1\}$.

A bi-ideal A of a semigroup S is called *two-sided pure* (or shortly *T-pure*) if

$$A \cap xSy = xAy$$

holds for every couple x, y of elements of S. A semigroup S is said to be *T^*-pure* if every bi-ideal of S is T-pure (see [1]). N. Kuroki proved that

$$x^m A y^n = xAy$$

provided A is a bi-ideal of a T^*-pure semigroup S, $x, y \in S$, and m, n are positive integers such that $m, n \geqq 2$. This result implies the following one.

THEOREM 5. *Suppose that S is a T^*-pure semigroup, $m, n \geqq 2$. Then S is (m, n)-twin if and only if it is (1, 1)-twin.*

References

[1] KUROKI, N.; T^*-pure twin semigroups, *Comm. Math. Univ. St. Pauli* **25** (1976), 107–114.

[2] LAJOS, S.; Notes on (m, n)-ideals, *Proc. Japan Acad.* **39** (1963), 419–421.

[3] LAJOS, S.; Generalized ideals in semigroups, *Acta Sci. Math,,* **22** (1961), 217–222.

Department of Mathematics
Karl Marx University for Economics
H-1828, Budapest
Hungary